Corrigendum

November 2, 2018

Correction of the proof of Theorem 5.8

Assumption (M) in the proof of Theorem 5.8 is not correct. Recall that this proof is an induction on the complexity of formulas in the language of set theory, and that only the cases for = and $\in$ use the assumption (M). We will give a new proof of these two cases by simultaneous induction on the definition of elements of $M^{(H)}$ without relying on assumption (M).

By simultaneous induction we will show that for all $x, y \in M^{(H)}$:

$$\begin{aligned} [\![x = y]\!]^{(H)} &= \{v \in K \mid f_v(x) = f_v(y)\}, \\ [\![x \in y]\!]^{(H)} &= \{v \in K \mid f_v(x) \in f_v(y)(v)\}. \end{aligned}$$

Let us begin with the first case.

$$\begin{aligned} [\![x = y]\!]^{(H)} &= \bigwedge_{z \in \mathrm{dom}(y)} (y(z) \to [\![z \in x]\!]) \wedge \bigwedge_{z \in \mathrm{dom}(x)} (x(z) \to [\![z \in y]\!]) \\ &= \bigwedge_{z \in \mathrm{dom}(y)} (y(z) \to \{v \in K \mid f_v(z) \in f_v(x)(v)\}) \\ &\qquad \wedge \bigwedge_{z \in \mathrm{dom}(x)} (x(z) \to \{v \in K \mid f_v(z) \in f_v(y)(v)\}) \\ &= \bigcap_{z \in \mathrm{dom}(y)} \{w \in K \mid \forall v \geq w (v \in y(z) \to f_v(z) \in f_v(x)(v))\} \\ &\qquad \cap \bigcap_{z \in \mathrm{dom}(x)} \{w \in K \mid \forall v \geq w (v \in x(z) \to f_v(z) \in f_v(y)(v))\} \\ &= \{w \in K \mid f_w(x) = f_w(y)\}. \end{aligned}$$

The first equality holds by definition of equality in Heyting-valued models, the second equality by induction hypothesis and the third equality by the definition of the logical operations in Heyting algebras of upsets.

Let us give an argument for the final equality: The direction from bottom to top follows from the fact that the transition functions f_{wv} are restrictions with $f_{wv} \circ f_w = f_v$ (see also the discussion before Theorem 5.8). Assume that

$f_w(x) = f_w(y)$, then $f_v(x) = f_v(y)$ for all $v \geq w$. So if $v \in x(z)$ for $v \geq w$, then $f_v(z) \in f_v(x)(v) \subseteq f_v(y)(v)$ by definition of f_v and our assumption. Simultaneously, we can prove that if $v \in y(z)$ for $v \geq w$, then $f_v(y)(v) \in f_v(x)$.

For the other direction we need to show that $f_w(x) = f_w(y)$ assuming that $w \in \mathsf{LHS}$. We will prove the first direction and the second direction follows symmetrically. Let $z' \in f_w(x)(v)$ for some $v \geq w$, then by definition of f_w, $z' = f_v(z)$ for some $z \in \mathrm{dom}(x)$ with $v \in x(z)$. By $w \in \mathsf{LHS}$, it follows that $z' = f_w(z) \in f_w(y)(w)$. This completes the case for $=$.

To prove the case for $\in$ observe that:

$$\begin{aligned}
[\![x \in y]\!] &= \bigvee_{z \in \mathrm{dom}(y)} (y(z) \wedge [\![x = z]\!]) \\
&= \bigcup_{z \in \mathrm{dom}(y)} (y(z) \cap \{w \in K \mid f_w(x) = f_w(y)\}) \\
&= \{w \in K \mid \exists z \in \mathrm{dom}(y)(w \in y(z) \wedge f_w(x) = f_w(z))\} \\
&= \{w \in K \mid f_w(x) \in f_w(y)(w)\}.
\end{aligned}$$

The first equality is just the definition of $\in$, the second equality follows from the induction hypothesis and the definition of the logical operations in a Heyting algebra of upsets, the third equality is clear. We will give an argument for the last equality: If $w \in K$ such that there is some $z \in \mathrm{dom}(y)$ with $w \in y(z)$ and $f_w(x) = f_w(z)$, then, by $w \in y(z)$ have $f_w(z) \in f_w(y)(w)$ and hence, $f_w(x) \in f_w(y)(w)$. Conversely, if $w \in K$ such that $f_w(x) \in f_w(y)(w)$, then, by definition of f_w, this means that there is some $z \in \mathrm{dom}(y)$ such that $w \in y(z)$ and $f_w(z) = f_w(x)$ and this is exactly what we need.

This finishes the $\in$-case and finishes the proof of our claim. Now note that, by the definition of the semantics in Lubarsky models, we have:

$$\begin{aligned}
[\![x = y]\!]^{(H)} &= \{v \in K \mid f_v(x) = f_v(y)\} = \{v \in K \mid v \Vdash f_v(x) = f_v(y)\}, \\
[\![x \in y]\!]^{(H)} &= \{v \in K \mid f_v(x) \in f_v(y)(v)\} = \{v \in K \mid v \Vdash f_v(x) \in f_v(y)\},
\end{aligned}$$

and this finishes the proof of the base cases of Theorem 5.8.